On the flow of granular materials with variable material properties

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Abstract

The mechanics of flowing granular materials such as coal, sand, agricultural products, fertilizers, dry chemicals, metal ores, etc., and their flow characteristics have received considerable attention in recent years. In this paper, the governing equations for the flow of granular materials are derived using a continuum approach. For a fully developed flow of these materials down an inclined plane, the equations reduce to a system of coupled non-linear ordinary differential equations for the case where the material properties are assumed to vary quadratically with the volume fraction. The boundary value problem is solved numerically and the results are presented for the volume fraction and velocity profiles.

Keywords: Continuum mechanics; Granular materials; Inclined flow; Steady fully developed flow

1. Introduction

A recent study indicates that the commercial and large-scale solids-processing plants have an average operating reliability of 63%, compared to 84% for large-scale plants using only liquids and gases (cf. Ref. [1]). Such a poor understanding of the flow of granular materials has serious economic consequences. Because we cannot yet reliably scale up laboratory or pilot-plant designs to commercial sizes, engineers are forced to resort to costly, cut-and-try methods of design. Also a major challenge facing the designers of coal gasification plants is to assure reliable and efficient movement of solids into and out of high-pressure, high-temperature fluidized-bed processing units. Earlier studies of the flow of granular material were mainly concerned with the engineering and structural design of bins and silos. The inaccuracy of these theories, especially for dynamic conditions of loading or emptying, occasionally resulted in failure of the bin or silo (cf. Ref. [2]). Also many situations such as discharge through bin outlets, flow through hoppers and chutes (cf. Ref. [3]), flow in mixers, and slurry transports require information on the flow patterns (cf. Refs. [4,5,59]). Therefore, to design equipment such as bins and silos, combustors, hoppers, chutes, hydrocyclones, etc., in an effective and economical way, a thorough understanding of the various factors governing the flow characteristics of granular materials must be obtained. These design needs have already motivated extensive analytical and experimental investigations of the flow of granular

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Nomenclature

$\mathbf{b}$	body force vector
$\mathbf{D}$	symmetric part of the velocity gradient
$\mathbf{F}$	deformation gradient
g	acceleration due to gravity
h	characteristic length
$\mathbf{1}$	identity tensor
$\mathbf{L}$	gradient of the velocity vector
N	dimensionless average volume fraction
t	time
$\mathbf{T}$	Cauchy stress tensor
u_0	reference velocity
$\bar{U}$	dimensionless velocity of the granular material
$\mathbf{x}$	spatial position occupied at time t
$\mathbf{X}$	reference position
y	direction normal to the inclined plane
(Y) or $\bar{y}$	dimensionless y
α	angle of inclination of the inclined plane with the horizontal
β_i	granular material constitutive coefficients, $i = 0$–4
χ	deformation function
γ	distributed mass density
ν	volume fraction
ρ	bulk density
ρ_0	reference density
div	divergence operator
∇	gradient symbol
$\otimes$	outer product

materials. Despite wide interest and more than five decades of experimental and theoretical investigations many aspects of the behavior of flowing granular materials are still not well understood. At this stage, however, there is no clear understanding of the constitutive relations that govern the flow of granular materials. The general field is still in a stage of development comparable to that of fluid mechanics before the advent of the Navier–Stokes relations.

Granular materials exhibit both the properties of a solid and a fluid as they can take the shape of the vessel containing them, thereby exhibiting fluidlike characteristics, or they can be heaped, thereby behaving like a solid. Also, granular materials can sustain shear stresses in the absence of any deformation, and the critical stress at which shearing begins depends on the normal stress. The characteristics of the particles that constitute the bulk solids are probably of major importance in influencing the characteristics of the bulk solids both at rest and during flow. It is very difficult to characterize bulk solids, which are composed of a variety of materials mainly because small variations in some of the primary properties of the bulk solids, such as the size, shape, hardness, particle density, and surface roughness can result in very different behavior. Furthermore, secondary factors such as the presence or absence of moisture, the severity of prior compaction, the ambient temperature, etc., which are not directly associated with the particles, can have a significant effect on the behavior of the bulk solids. A granular material covers the combined range of granular powders and granular solids with components ranging in size from about 10 μm up to 3 mm. A powder is composed of particles up to 100 μm (diameter) with further subdivision into ultrafine (0.1–1.0 μm), superfine (1–10 μm), or granular (10–100 μm) particles. A granular solid consists of materials ranging from about 100 to 3000 μm (cf. Ref. [4]).

Also, little is known of the relationship between particle shape and flow properties in detail although it is observed that smooth spherical particles display more favorable flow conditions than particles with a sharp angular surface, especially if they have a tendency to interlock. In addition, moisture content of the bulk solids is one of the most important factor controlling the flow properties of the granular materials. In fact, moisture content in bulk solids is mostly undesirable, because the surface moisture leads to the appearance of cohesive forces between particles and of adhesive forces between particles and the walls of the container. Both retard the flow of solid particles and under certain conditions may even stop the flow entirely. Since for the same weight the total surface of solids is greater for smaller grains, the surface moisture content increases inversely as the particle diameter. Therefore, fine particles display more cohesive and adhesive forces than the larger grains. Furthermore, fine particles when stored for a certain time undisturbed, have a tendency to compact, that is to reduce the total volume which creates additional resistance to the flow. In general, the flow properties of most materials can be expected to change drastically as moisture content changes, particularly for finer materials (cf. Ref. [6]).

Due to their complexity, the modeling of granular materials would require a fusion of the ideas from solid, fluid, and soil mechanics. Granular materials, like non-Newtonian fluids (cf. Ref. [7]) and non-linearly elastic solids, exhibit normal-stress differences in simple shear flow. Thus, modeling granular materials and slurries is very complex and has to draw upon experiences from non-linear fluid and solid theories. Scientific understanding of the flow of granular materials has been hampered both by the difficulty of making measurements and by a tendency to look for immediate engineering solutions to individual problems as they arise. The flow of granular materials strongly depends upon the distribution of void space. Experiments have to be devised to quantify and describe the non-linear behavior of such materials, and theories have to be developed to explain the experimentally observed facts and predict other qualitative phenomena confirmable by further experiments.

In the past several years many researchers have tried to address some of these issues. Rajagopal and Massoudi [8] have outlined an experimental/theoretical procedure to measure the material properties in an orthogonal rheometer. The model proposed by them, which is based on the works of Cowin and Savage, has been used to study various problems such as flow in a vertical pipe (cf. Ref. [9]), heat transfer and flow on an inclined plane [10], flow due to natural convection (cf. Ref. [11]). At the same time this model has been used within the context of mixture theory to study problems of practical interest in multiphase applications (cf. Ref. [12]). Flow down an inclined plane has been studied extensively by many researchers. Recent review articles by Campbell [13], Hutter and Rajagopal [14], and de Gennes [15] address many of the interesting issues in the filed of granular materials. In this paper, we extend the result of Gudhe et al. [10] to the case where the material properties are functions of volume fraction.

2. Governing equations

The balance laws, in the absence of chemical reactions and thermal effects, are the conservation of mass, conservation of linear momentum, and conservation of angular momentum. Conservation of mass in the Lagrangian form is

$$\rho_0 = \rho \det \mathbf{F}, \tag{1}$$

where, ρ_0 is the reference density of the material, ρ is the current density, and $\mathbf{F}$ is the deformation gradient which is given by

$$\mathbf{F} = \frac{\partial \chi}{\partial \mathbf{X}}. \tag{2}$$

The conservation of mass in the Eulerian form is given by

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0, \tag{3}$$

where $\partial/\partial t$ is the partial derivative with respect to time. The balance of linear momentum is

$$\rho \frac{D\mathbf{u}}{Dt} = \operatorname{div} \mathbf{T} + \rho \mathbf{b}, \tag{4}$$

where D/D is the material time derivative, $\mathbf{b}$ is the body force, and $\mathbf{T}$ is the Cauchy stress tensor. The balance of angular momentum (in the absence of couple stresses) yields the result that the Cauchy stress is symmetric.

3. Constitutive equation

Early experimental investigations of granular materials were conducted by Hagen [16] who studied the flow of sand in tubes. Reynolds [17] observed that for a shearing motion to occur in a bed of closely packed particles, the bed must expand to increase the volume of its voids. He termed this phenomena "*dilatancy*", [cf. Ref. [18]]. Reynolds [19] used the idea of "dilatancy" to describe the capillary action in wet sand. Many of the existing theories for flowing granular materials use this observation to relate the applied stress to the voidage and the velocity. Later, Reiner [20] proposed a continuum model to describe the mechanics of wet sand. This model does not take into account how the voidage (volume fraction) affects the stress. However, using his model, Reiner [20] showed that application of a non-zero shear stress produces a change in volume. McTigue [21] discusses the extension of the Reiner–Rivlin model to granular materials.

Bagnold [22] performed experiments on neutrally buoyant, spherical particles suspended in Newtonian fluids undergoing shear in coaxial rotating cylinders. He was able to measure the torque and normal stress in the radial direction for various concentrations of the grains. He distinguished three different regimes of flow behavior, which he termed macro-viscous, transitional, and grain-inertia. In the so-called "macro-viscous" region, which corresponds to low shear rates, the shear and normal stresses are linear functions of the velocity gradient. In this region, the fluid viscosity is the dominant parameter. In the region, called the "grain-inertia region", the fluid in the interstices does not play an important role and the dominant effects arise from particle–particle interactions. Here, the shear and the normal stresses are proportional to the square of the velocity gradient. Connecting the two limiting flow regimes was the Bagnold's transitional flow, in which the dependence of the stress on shear rate varied from a linear one corresponding to the macro-viscous regime to a square dependence predicted for the grain-inertia flow regime. The interesting phenomenon was the presence of a normal stress proportional to the shear stress, similar to that of the quasi-static behavior of a cohesionless material obeying the Mohr–Coulomb criterion. From his experiments, Bagnold was able to define the various flow regimes in terms of dimensionless number $\bar{N}$, later referred to as the Bagnold number, given by

$$\bar{N} = \lambda_0^{1/2} \rho_f \bar{\sigma}^2 \frac{(u_{1,2})}{\mu_f}. \tag{5}$$

Here ρ_f and μ_f are the mass density and viscosity of the fluid, $\bar{\sigma}$ is the diameter of the particle, λ_0 is the linear concentration of particles, and $u_{1,2}$ is the velocity gradient. The macro-viscous regime corresponds to $\bar{N} < 40$, and the grain-inertia regime to $\bar{N} > 450$. He called the intermediate range of $\bar{N}$ the "transitional" region. Bagnold [22] applied his cylindrical shear cell results and his analysis for the stresses to study the problems of gravity flow of particulate matter down inclines as might occur in rock falls and debris flows (cf. Ref. [23]).

Another criterion often used when devising a theory for the flow of granular materials is that the equilibrium states specified by the theory are required to coincide with the limiting equilibrium states specified by the Mohr–Coulomb criterion. The Coulomb failure criterion (cf. Refs. [24,25]), based on experiments, states that yielding will occur when

$$S = b_0 T + c. \tag{6}$$

Here S and T are the shear stress and normal stress, respectively, acting on a plane at a point; c is the coefficient of cohesion, and b_0 is the coefficient of static friction related to the internal angle of friction ϕ through

$$b_0 = \tan\phi. \tag{7}$$

When cohesion is absent ($c = 0$), it is usual to call a granular medium an ideal one. One in which internal friction is absent ($\phi = 0$) is called an ideally cohesive medium. For dry, coarse materials, the cohesion coefficient can be neglected. Typical values for the internal angle of friction, ϕ, obtained during quasi static yielding at low stress levels are close to the angles of repose, e.g., about 24° for spherical glass beads and 38° for angular sand grains (cf. Ref. [4]).

The pioneering work of Reynolds and Bagnold was followed by several others who attempted to model the mechanics of granular materials. These approaches can be classified under two general categories: statistical and continuum theories. The statistical theories which include various versions of the kinetic theory of gases, turbulence models, particle simulation (cf. Ref. [61]), etc. are not discussed here. In the continuum approach it is assumed that the material properties of the ensemble may be represented by continuous functions so that the medium may be divided infinitely without losing any of its defining properties. Continuum plasticity-type models based on the Mohr–Coulomb criterion for failure have been proposed by Drucker and Prager [26], Spencer [27,60], and Jenike [28].

One of the early continuum models for flowing granular materials based on the principles of modem continuum mechanics was proposed by Goodman and Cowin [29,30]. They used the ideas that had already been developed for materials with microstructure (or oriented materials) such as liquid crystals and micropolar materials. Cowin [31,32] and Savage [33] proposed a non-linear theory for incompressible granular materials, which represents the flow of granular materials at relatively low stress levels and high shear rates such that the bulk behavior of the material is primarily due to interparticle friction. Cowin [32] showed that by including the gradient of the volume fraction as one of the important parameters in proposing a constitutive equation for the stress tensor, a theory can be devised for the flow of granular materials. In this theory a critical yield condition called the Mohr–Coulomb emerges naturally, as does the transition between the frictional flow regimes, characterized by the absence of deformation and the viscous flow regime, characterized by deformation.

The Cauchy stress tensor $\mathbf{T}$ in a flowing granular material may depend on the manner in which the granular material is distributed, i.e., the volume fraction v and possibly also its gradient, and the symmetric part of the velocity gradient tensor $\mathbf{D}$. Thus, we assume that

$$\mathbf{T} = \mathbf{f}(v, \nabla v, \mathbf{D}). \tag{8}$$

Using standard arguments in mechanics, restrictions can be found on the form of the above constitutive expression based on the assumption of frame-indifference, isotropy, etc. (cf. Ref. [34]). There could be further restrictions on the form of the constitutive expression because of internal constraints, such as, incompressibility and thermodynamics restrictions due to Clausius–Duhem inequality (cf. Ref. [35]). A constitutive model that predicts the possibility of one normal stress-difference and its properly frame invariant is given by (cf. Ref. [8]):

$$\mathbf{T} = \{\beta_0(v) + \beta_1(v)\nabla v.\nabla v + \beta_2(v)\mathrm{tr}\,\mathbf{D}\}\mathbf{1} + \beta_4(v)\nabla v\otimes\nabla v + \beta_3(v)\mathbf{D}, \tag{9}$$

where the following interpretation can be given to the material parameters: $\beta_0(v)$ is similar to pressure in a compressible fluid and is given by an equation of state, $\beta_2(v)$ is like the second coefficient of viscosity in a compressible fluid, $\beta_1(v)$ and $\beta_4(v)$ are the material parameters connected with the distribution of the granular materials, and $\beta_3(v)$ is the viscosity of the granular materials. The above model allows for normal stress differences, a feature observed in granular materials (cf. Ref. [36]). In general, the material properties β_0 through β_4 are functions of the density (*or* volume fraction v), temperature, and the principal invariants of the tensor $\mathbf{D}$, given by

$$\mathbf{D} = \tfrac{1}{2}[(\nabla\mathbf{u}) + (\nabla\mathbf{u})^{\mathrm{T}}], \tag{10}$$

where $\mathbf{u}$ is the velocity of the particles. In Eq. (9), $\mathbf{1}$ is the identity tensor, ∇ the gradient operator, $\otimes$ indicates the outer (dyadic) product of two vectors, and *tr* designates the trace of a tensor. Furthermore, v is related to the bulk density of the material ρ, through

$$\rho = \gamma v, \tag{11}$$

where γ is the actual density of the grains at the place $\mathbf{x}$ and time t and the field v is called the volume fraction (*or* the volume distribution) and is related to the porosity $\bar{n}$ or the void ratio $\bar{e}$ by

$$v = 1 - \bar{n} = \frac{1}{1+\bar{e}} \quad \text{with } 0 \leqslant v < 1. \tag{12}$$

It was mentioned earlier that $\beta_0(v)$ plays the role of pressure in a compressible gas, with v now playing the role of the density. The works of Walton and Braun [37,38] assume that the viscosity is a function of both the solid fraction v and the tensor $\mathbf{D}$, and varies as a quadratic function of v, $\mathbf{D}$ being held fixed. Following, Rajagopal and Massoudi [8] we assume that the material parameters have the structure

$$\begin{aligned}
\beta_0(v) &= kv,\\
\beta_1(v) &= \beta_{10} + \beta_{11}v + \beta_{12}v^2,\\
\beta_2(v) &= \beta_{20} + \beta_{21}v + \beta_{22}v^2,\\
\beta_3(v) &= \beta_{30} + \beta_{31}v + \beta_{32}v^2,\\
\beta_4(v) &= \beta_{40} + \beta_{41}v + \beta_{42}v^2.
\end{aligned} \tag{13}$$

The above representation can be viewed as a Taylor series approximation for the material parameters. Such a quadratic dependence, at least for the viscosity β_3, is on the basis of dynamic simulations of particle interactions (cf. Refs. [37,38]). Further restrictions on the coefficients can be obtained by using the following argument. Since the stress should vanish as $v \to 0$, we can conclude that

$$\beta_{30} = \beta_{20} = 0. \tag{14}$$

This is really a principle of the limiting case. That is, if there are no particles, then v and grad v are zero. And when there are no particles, the stress should also be zero; however, the kinematical terms $\mathbf{D}$ and *tr* $\mathbf{D}$, multiplied by β_2 and β_3 in Eq. (9) do not necessarily go to zero when there are no particles. Therefore, to ensure this we impose the restriction given by Eq. (14). The rationale for the structure given above can also be found in Ref. [8]. Also, Johnson et al. [39,40] have used this model to study two-phase flows. Furthermore, Rajagopal and Massoudi [8] and Rajagopal et al. [41] have shown that

$$k < 0 \tag{15}$$

as compression should lead to densification of the material.

Eq. (9) represents a general constitutive relation for flowing granular materials. The material parameters $\beta_0 - \beta_4$ have to be specified before solving a boundary value problem. Rajagopal and Massoudi [8] and Rajagopal et al. [42] outlined an experimental procedure where they showed that using an orthogonal rheometer, these material parameters, in principle, can be measured. An alternative way of deriving exact forms for these (rheological) properties is to use another theory such as the kinetic theory of gases, or a statistical theory, where the explicit dependence of these material parameters on other primary variables such as particle diameter, particle volume fraction, fluctuation of particles, particle distribution, etc., can be obtained. Boyle and Massoudi [43] have provided this information for this particular model. Furthermore, many granular materials exhibit a yield stress before they begin to flow. A typical yield criterion used very frequently in granular materials literature is the Mohr–Coulomb criterion. Cowin [32] and Savage [2] have shown that a similar constitutive relation to Eq. (9) is capable of complying with the Mohr–Coulomb criterion if a specific representation is given to β_0, relating it to the internal angle of friction.

4. Numerical solution

The flow of granular materials down an inclined plane has been studied by several authors (cf. Refs. [2,29,44–46,58]). Hutter et al. [44,45] show that the existence or non-existence of solutions to their equations depend on the type of boundary conditions that they impose. Recently, Rajagopal et al. [42] also studied the existence and uniqueness of solutions to the equations governing the flow of granular materials down an inclined plane. (The constitutive relation used by them is the same as Eq. (9) which is different from the one used by Hutter et al. [44,45]). They delineate a range of values for the material parameters, which are assumed to be constant and ensure existence of solutions to the equations under consideration. They also prove that for certain range of values of the material parameters no solution exists, while for a different range there is multiplicity of solutions.

In this problem, we consider steady one-dimensional fully developed flow of incompressible granular materials (i.e., $\gamma = \text{constant}$) down an inclined plane, where the angle of inclination is α. Furthermore, the volume fraction and the velocity fields are assumed to be of the form

$$v = v(y),$$

$$u = U(y). \tag{16}$$

It is assumed that β_0 is given by Eqs. (13) and (15) with β_1, β_2, β_3, and β_4 to be quadratic in volume fraction as given by

$$\beta_1 = \hat{\beta}_1(1 + v + v^2), \tag{17}$$

$$\beta_2 = \hat{\beta}_2(1 + v + v^2), \tag{18}$$

$$\beta_3 = \hat{\beta}_3(v + v^2), \tag{19}$$

$$\beta_4 = \hat{\beta}_4(1 + v + v^2), \tag{20}$$

where, $\hat{\beta}_1$–$\hat{\beta}_4$ are constants. With the above assumptions and the flow field given by Eq. (16), the conservation of mass is identically satisfied and the balance of linear momentum reduces to

$$k\frac{\mathrm{d}v}{\mathrm{d}y} + 2(\hat{\beta}_1 + \hat{\beta}_4)(1 + v + v^2)\frac{\mathrm{d}v}{\mathrm{d}y}\frac{\mathrm{d}^2 v}{\mathrm{d}y^2} + (\hat{\beta}_1 + \hat{\beta}_4)(1 + 2v)\left\{\frac{\mathrm{d}v}{\mathrm{d}y}\right\}^3 = \gamma g v \cos\alpha, \tag{21}$$

$$\hat{\beta}_3(v + v^2)\frac{\mathrm{d}^2 U}{\mathrm{d}y^2} + \hat{\beta}_3(1 + 2v)\frac{\mathrm{d}v}{\mathrm{d}y}\frac{\mathrm{d}U}{\mathrm{d}y} = -2\gamma g v \sin\alpha. \tag{22}$$

In general, whether we use the kinetic theory approach (cf. Refs. [47,54,57]) or the continuum approach to solve boundary value problems, the need for additional boundary conditions arises. In the continuum theories of Goodman and Cowin [29,30] and its modifications (cf. Refs. [39,40,48,49, 55,56]), two boundary conditions on the volume fraction are required. In the numerical solution of shearing motion of a fluid–solid flow, Passman et al. [49] prescribed the values of the volume fraction at the two plates. An alternative way is to use experimental results, if they are available (cf. Ref. [53]). Later Johnson et al. [39,40] considered this issue and suggested using an integral condition for the value of volume fraction. Looking at Eqs. (21) and (22) it is clear that we need two boundary conditions for the volume fraction, and two boundary conditions for the velocity.

Eqs. (21) and (22) are to be solved subject to the following boundary conditions:

$$U = 0 \quad \text{at } y = 0 \quad \text{(on the inclined plane)} \tag{23}$$

and,

$$\frac{\mathrm{d}U}{\mathrm{d}y} = 0, \tag{24a}$$

$$kv + (\hat{\beta}_1 + \hat{\beta}_4)(1 + v + v^2)\left\{\frac{\mathrm{d}v}{\mathrm{d}y}\right\}^2 = 0 \quad \text{at } y = h$$

(at the free surface) (24b)

and the constraint that

$$Q_0 = \int_0^h v\,\mathrm{d}y, \quad Q_0 \text{ being given.} \tag{25}$$

as the ratio of the pressure force to the gravity force, A_2 is the ratio of forces developed in the material due to the distribution of the voids to the force of gravity, and A_3 is the ratio of the viscous force to the gravity force (related to the Reynolds number).

The system of equations (27) and (28) with the boundary conditions (29)–(31) and, subject to the restriction (16), are solved numerically using a collocation code COLSYS (cf. Ref. [50]). It follows from Rajagopal and Massoudi [8] that R_1 must always be less than zero for the solution to exist and all the other non-dimensional parameters, i.e., A_2 and A_3 must be greater than zero. A parametric study of the equations is carried out to see how the various non-dimensional parameters affect the volume fraction and the velocity profiles.

The manner in which the volume fraction and the velocity profiles change with R_1, is shown in Figs. 1 and 2, respectively. Notice, that the volume fraction profile decreases from the surface of the plane to the free surface, which is to be expected. Increasing the magnitude of R_1, with the other constants being held fixed, results in a decrease of velocity. Increasing values of A_2 results in a decrease of volume fraction and increase in velocity (cf. Figs. 3 and 4). Fig. 5 shows the effect of A_3 on the velocity profile. Notice that as A_3 increases the velocity decreases, which is expected because A_3 is the inverse of the Reynolds number.

Finally, we can obtain an exact solution to the system of equations (21) and (22). If we assume that β_0 is zero and β_1–β_4 are constants, then the momentum equation in the y-direction, i.e., Eq. (21) can be integrated directly to give an expression for the volume fraction:

$$\nu = \frac{F}{18}y^3 + \left(\frac{F^2\nu_0}{12}\right)^{1/3} y^2 + \left(\frac{3F\nu_0^2}{2}\right)^{1/3} y + \nu_0, \quad (33)$$

where

$$F = \frac{\gamma g \cos\alpha}{2(\beta_1 + \beta_2)} \quad (34)$$

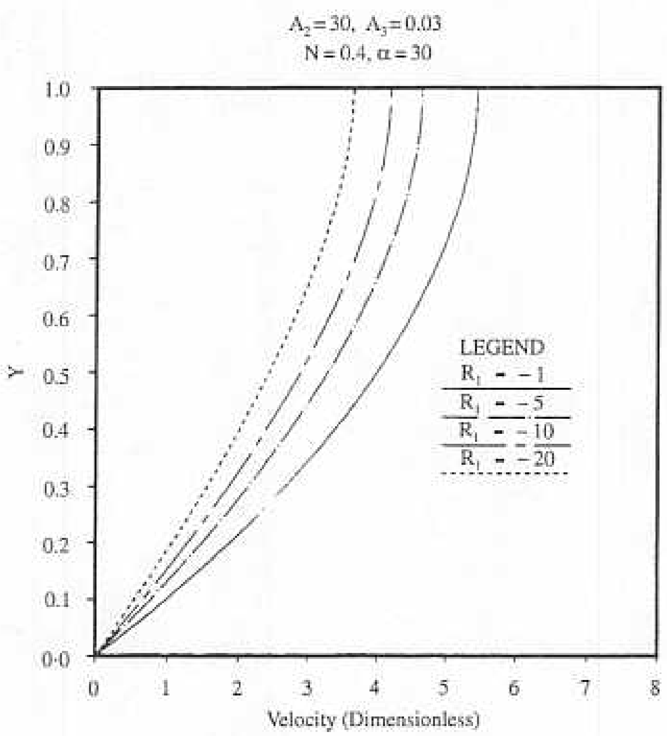

Fig. 2. Effect of R_1 on the velocity profile.

Also, notice that Eqs. (24a) and (24b) are the stress-free conditions, Eq. (23) indicates the no-slip condition (rough wall) assumption. Now, the system of Eqs. (21) and (22) subject to the boundary conditions (23), (25) are non-dimensionalized using the following equations:

$$\bar{y} = \frac{y}{h}, \quad \bar{u} = \frac{u}{u_0}. \tag{26}$$

The above system of equations reduces to

$$R_1 \frac{\mathrm{d}v}{\mathrm{d}\bar{y}} + A_2(1 + v + v^2)\frac{\mathrm{d}v}{\mathrm{d}\bar{y}}\frac{\mathrm{d}^2 v}{\mathrm{d}\bar{y}^2} + \frac{A_2}{2}(1 + 2v)\left\{\frac{\mathrm{d}v}{\mathrm{d}\bar{y}}\right\}^3 = v\cos a, \tag{27}$$

$$A_3 v(1 + v)\frac{\mathrm{d}^2\bar{U}}{\mathrm{d}\bar{y}^2} + A_3(1 + 2v)\frac{\mathrm{d}v}{\mathrm{d}\bar{y}}\frac{\mathrm{d}\bar{U}}{\mathrm{d}\bar{y}} = -v\sin a \tag{28}$$

and the boundary conditions become

$$\bar{U} = 0, \quad \text{at } \bar{y} = 0 \quad \text{(on the inclined plane)}, \tag{29}$$

$$N = \int_0^1 v\,\mathrm{d}\bar{y} \tag{30}$$

and,

$$\frac{\mathrm{d}\bar{U}}{\mathrm{d}\bar{y}} = 0, \tag{31a}$$

$$R_1 v + \frac{A_2}{2}(1 + v + v^2)\left\{\frac{\mathrm{d}v}{\mathrm{d}\bar{y}}\right\}^2 = 0 \tag{31b}$$

at $\bar{y} = 1$ (at the free surface)

The non-dimensional parameters R_1, A_2 and A_3 are given by

$$R_1 = \frac{k}{h\gamma g}, \quad A_2 = \frac{2(\hat{\beta}_1 + \hat{\beta}_4)}{h^3\gamma g}, \quad A_3 = \frac{\beta_3 U_0}{2h^2\gamma g}. \tag{32}$$

These dimensionless parameters have the following physical interpretations: R_1 could be thought of

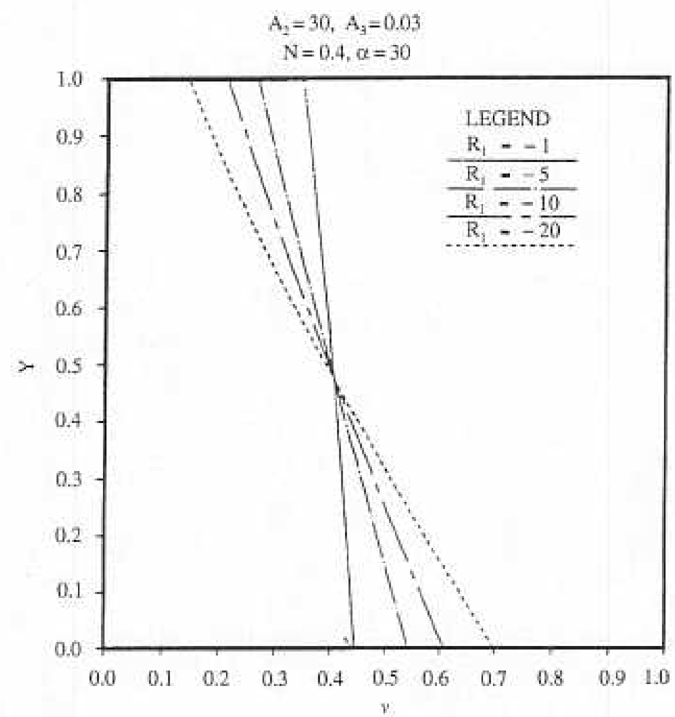

Fig. 1. Effect of R_1 on the volume fraction.

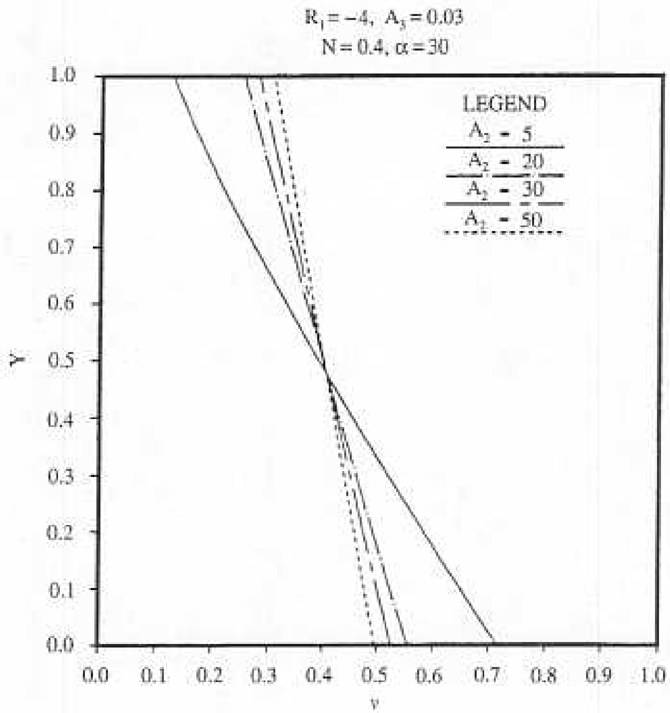

Fig. 3. Effect of A_2 on the volume fraction.

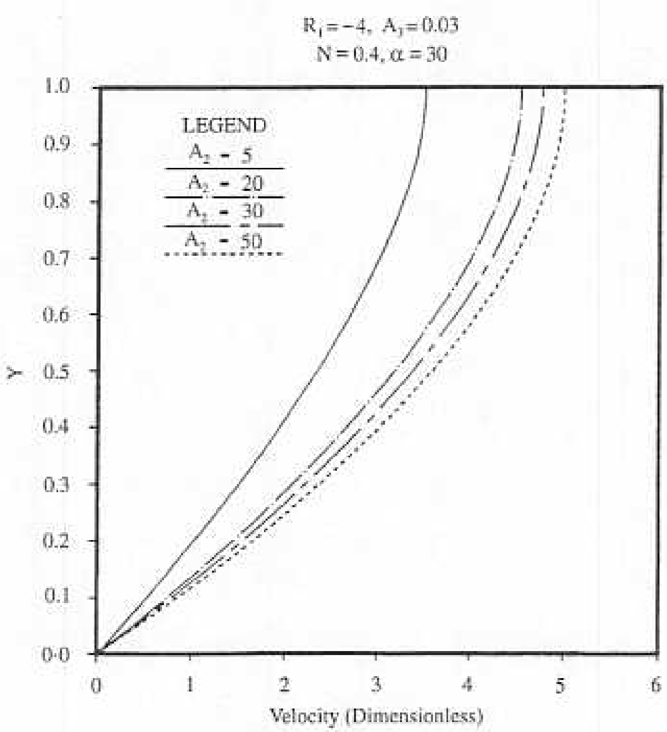

Fig. 4. Effect of A_2 on the velocity profile.

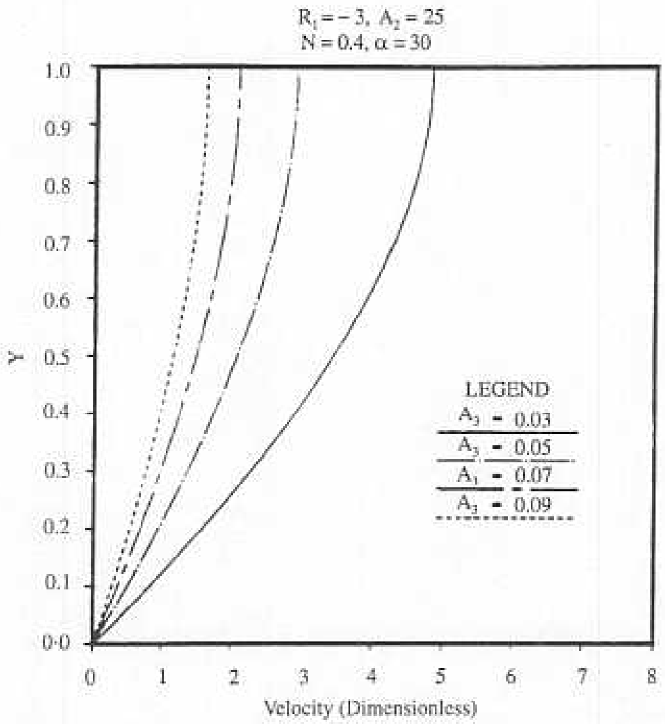

Fig. 5. Effect of A_3 on the velocity profile.

and ν_0 is a specified value for the volume fraction at the plate. Once we substitute this expression into the x-component of the momentum equation and integrate, we obtain an explicit relation for the velocity of particles:

$$u = -\frac{2\gamma g \sin\alpha}{\beta_3}\left\{\frac{F}{360}y^5 + \frac{1}{12}\left(\frac{F^2\nu_0}{12}\right)^{1/3} y^4 + \frac{1}{6}\left(\frac{3F\nu_0^2}{2}\right)^{1/3} y^3 + \frac{\nu_0}{2}y^2\right\} + \frac{2\gamma g \sin\alpha}{\beta_3}\left\{\frac{F}{72}h^4 + \frac{1}{3}\left(\frac{F^2\nu_0}{12}\right)^{1/3} h^3 + \frac{1}{2}\left(\frac{3F\nu_0^2}{2}\right)^{1/3} h^2 + \nu_0 h\right\} y, \tag{35}$$

where h is the distance from the plate to the free surface, assumed to be known. In obtaining Eqs. (33) and (35), a no-slip boundary condition is used at the wall ($y = 0$), and a traction free condition is imposed at $y = h$.

5. Conclusions

The main reason for doing a parametric study, via non-dimensionalizing the equations of motion, is that we can gain some insight into a class of problems. Since the material parameters β_0–β_4 have not been measured experimentally, it is not possible to compare our results to any experiment, quantitatively . However, qualitatively we can see that since the material parameters are functions of the volume fraction, there is a stronger non-linearity in the equations, and therefore, numerically it is more difficult to obtain solutions. The case where β_1 and β_4 are constants, was studied by Gudhe et al. [10]. One of the effects that we observe in the present investigation is that the volume fraction profiles as depicted in Fig. 1 are influenced significantly by the variations in R_1 due to the presence of the higher-order terms in Eq. (27) and as a result, the velocity profiles are affected accordingly, due to the coupling of the Eqs. (27) and (28). Furthermore, because of the kinematical constraints imposed by Eq. (16), certain non-linear phenomenon, similar to the "hydraulic jump", which is observed in some experimental chute flows of granular materials (cf. Refs. [51,52]) cannot be obtained using the present formulation.

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References

[1] S.I. Plasynski, W.C. Peters, S.L. Passman, The Department of Energy solids transport, multiphase flow program, Proceedings of the NSF-DOE Joint workshop on Flow of Particulates and Fluids, Gaithersburg, September 16–18, 1992.
[2] S.B. Savage, Gravity flow of cohesionless granular materials in chutes and channels, J. Fluid Mech. 92 (1979) 53–96.
[3] R. Jyotsna, K. Kesava Rao, A frictional-kinetic model for the flow of granular materials through a wedge-shaped hopper, J. Fluid Mech. 346 (1997) 239–270.
[4] R.L. Brown, J.C. Richards, Principles of Powder Mechanics, Pergamon Press, London, 1970.
[5] R.D. Marcus, L.S. Leung, G.E. Klinzing, F. Rizk, Pneumatic Conveying of Solids, Chapman&Hall, London, 1990.
[6] A.J. Stepanoff, Gravity Flow of Bulk Solids and Transportation of Solids in Suspension, Wiley, New York, 1969.
[7] S.A. Elaskar, L.A. Godoy, Constitutive relations for compressible granular materials using non-Newtonian fluid mechanics, Int. J. Mech. Sci. 40 (1998) 1001–1018.
[8] K.R. Rajagopal, M. Massoudi, A method for measuring material moduli of granular materials: flow in an orthogonal rheometer, Topical Report, DOE/PETC/TR-90/3, 1990.
[9] R. Gudhe, R.C. Yalamanchili, M. Massoudi, Flow of granular materials down a vertical pipe, Int. J. Non-Linear Mech. 29 (1994) 1–12.
[10] R. Gudhe, K.R. Rajagopal, M. Massoudi, Heat transfer and flow of granular materials down an inclined plane, Acta Mech. 103 (1994) 63–78.
[11] M. Massoudi, T.X. Phuoc, Flow and heat transfer due to natural convection in granular materials, Int. J. Non-Linear Mech. 34 (1999) 347–359.
[12] M. Massoudi, K.R. Rajagopal, T.X. Phuoc, On the fully developed flow of a dense particulate mixture in a pipe, Powder Technol. 104 (1999) 258–268.
[13] C. Campbell, Rapid granular flows, Ann. Rev. Fluid Mech. 22 (1990) 57–92.
[14] K. Hutter, K.R. Rajagopal, On the flows of granular materials, Continuum Mech. Thermodyn. 6 (1994) 81–139.
[15] P.G. de Gennes, Reflections on the mechanics of granular matter, Physica A 261 (1998) 267–293.
[16] G. Hagen, Berlin Monatsber Akad. Wiss. 1852, pp. 35–42.
[17] O. Reynolds, On the dilatancy of media composed of rigid particles in contact with experimental illustrations, Philos. Mag. Ser. 5 (20) (1885) 469–481.
[18] J.D. Goddard, Granular dilatancy and the plasticity of glassy lubricants, Ind. Eng. Chem. Res. 38 (1999) 820–822.
[19] O. Reynolds, Experiments showing dilatancy, a property of granular material, possibly connected with gravitation, Proc. Roy. Inst. Gr. Britain 11 (1886) 354–363.
[20] M. Reiner, A Mathematical theory of dilatancy, Amer. J. Math. 67 (1945) 350–362.
[21] D.F. McTigue, A non-linear constitutive model for granular materials: applications to gravity flow, J. Appl. Mech. 49 (1982) 291–296.
[22] R.A. Bagnold, Experiments on a gravity free dispersion of large solid spheres in a Newtonian fluid under shear, Proc. Roy. Soc. London 225 (1954) 49.
[23] R.A. Bagnold, The flow of cohesionless grains in fluids, Philos. Trans. Roy. Soc. London, Ser. A 249 (1956) 235.
[24] V.V. Sokolovski, Statics of Granular Media, Pergamon, Oxford, 1965.
[25] R.M. Nedderman, Statics and Kinematics of Granular Materials, Cambridge University Press, Cambridge, 1992.
[26] D.C. Drucker, W. Prager, Soil mechanics and plastic analysis or limit design, Quart. Appl. Math. 10 (1952) 157–165.
[27] A.J.M. Spencer, A theory of the kinematics of ideal solids under plane strain conditions, J. Mech. Phys. Solids 12 (1964) 337–351.
[28] A.W. Jenike, Steady gravity flow of frictional-cohesive solids in converging channels, J. Appl. Mech. 31 (1964) 5–11.
[29] M.A. Goodman, S.C. Cowin, Two problems in the gravity flow of granular materials, J. Fluid Mech. 45 (1971) 321–339.
[30] M.A. Goodman, S.C. Cowin, A continuum theory for granular materials, Arch. Rational Mech. Anal. 44 (1972) 249–266.
[31] S.C. Cowin, A Theory for the Flow of Granular Material, Powder Technol. 9 (1974) 61–69.
[32] S.C. Cowin, Constitutive relations that imply a generalized Mohr–Coulomb criterion, Acta Mech. 20 (1974) 41–46.
[33] S.B. Savage, The mechanics of rapid granular flows, Adv. Appl. Mech. 24 (1984) 289–366.
[34] C. Truesdell, W. Noll, The non-linear field theories of mechanics, in: Flugge (Ed.), Handbuck Der Physik, III/I, Springer, Berlin, 1965.
[35] I. Müller, On the entropy inequality, Arch. Rational Mech. Anal. 26 (1967) 118–141.

[36] I. Goldhirsch, N. Sela, Origin of normal stress differences in rapid granular flows, Phys. Rev. E 54 (1996) 4458–4461.

[37] O.R. Walton, R.L. Braun, Stress calculations for assemblies of inelastic spheres in uniform shear, Acta Mech. 63/1-4 (1986) 73–86.

[38] O.R. Walton, R.L. Braun, Viscosity, granular-temperature, and stress calculations for shearing assemblies of inelastic, frictional disks, J. Rheol. 30 (1986) 949–980.

[39] G. Johnson, M. Massoudi, K.R. Rajagopal, Flow of a fluid-solid mixture between flat plates, Chem. Eng. Sci. 46 (1991) 1713–1723.

[40] G. Johnson, M. Massoudi, K.R. Rajagopal, Flow of a fluid infused with solid particles through a pipe, Int. J. Eng. Sci. 29 (1991) 649–661.

[41] K.R. Rajagopal, W.C. Troy, M. Massoudi, Existence of solutions to the equations governing the flow of granular materials, European J. Mech. B/Fluids 11 (1992) 265–276.

[42] K.R. Rajagopal, M. Massoudi, A.S. Wineman, Flow of granular materials between rotating disks, Mech. Res. Comm. 21 (1994) 629–634.

[43] E.J. Boyle, M. Massoudi, A theory for granular materials exhibiting normals stress effects based on Enskog's dense gas theory, Int. J. Eng. Sci. 28 (1990) 1261–1275.

[44] K. Hutter, F. Szidarovszky, S. Yakowitz, Plane steady shear flow of a cohesionless granular material down an inclined plane: a model for flow avalanches Part-I: theory, Acta Mech. 63 (1986) 87–112.

[45] K. Hutter, F. Szidarovszky, S. Yakowitz, Plane steady shear flow of a cohesionless granular material down an inclined plane: a model for flow avalanches Part-II: numerical results, Acta Mech. 65 (1986) 239–261.

[46] P.C. Johnson, R. Jackson, Frictional-collisional constitutive relations for granular materials with application to plane shearing, J. Fluid Mech. 176 (1987) 67–93.

[47] K. Hui, P.K. Haff, J.E. Ungar, R. Jackson, Boundary conditions for high-shear grain flows, J. Fluid Mech. 145 (1984) 223–233.

[48] G. Ahmadi, A generalized continuum theory for granular materials, Int. J. Non-Linear Mech. 17 (1982) 21–33.

[49] S.L. Passman, J.W. Nunziato, P.B. Bailey, K.W. Reed, Shearing motion of a fluid-saturated granular material, J. Rheol. 30 (1986) 167–192.

[50] U. Ascher, J. Christianson, R.D. Russel, Collocation software for boundary value ODE's, ACM Trans. Math. Software 7/2 (1981) 209–222.

[51] H. Ahn, C.E. Brennen, R.H. Sabersky, Analysis of the fully developed chute flows of granular materials, Trans. ASME 59 (1992) 109–119.

[52] H. Ahn, C.E. Brennen, R.H. Sabersky, Measurements of velocity fluctuation, density, and stresses in chute flows of granular materials, Trans. ASME 58 (1991) 792–803.

[53] K. Craig, R.H. Buckholz, G. Domoto, The effects of shear surface boundaries on stresses for the rapid shear of dry powders, ASME J. Tribol. 109 (1987) 232.

[54] J.T. Jenkins, S.B. Savage, A theory for the rapid flow of identical, smooth, nearly elastic, spherical particles, J. Fluid Mech. 130 (1983) 187–202.

[55] J.W. Nunziato, S.L. Passman, J.P. Thomas Jr., Gravitational flows of granular materials with incompressible grains, J. Rheol. 24 (1980) 395–420.

[56] S.L. Passman, J.W. Nunziato, P.B. Bailey, J.P. Thomas Jr., Shearing flows of granular materials, J. Eng. Mech. Div. ASCE 106 (1980) 773–783.

[57] M.W. Richman, R.P. Marciniec, Gravity-driven granular flows of smooth, inelastic spheres down bumpy inclines, Trans. ASME 57 (1990) 1036–1043.

[58] M. Sayed, S.B. Savage, Rapid gravity flow of cohesionless granular materials down inclined chutes, ZAMP 34 (1983) 84–100.

[59] S.L. Soo, Fluid Dynamics of Multiphase Systems, Blaisdell, New York, 1967.

[60] A.J.M. Spencer, Instability of steady shear flow of granular materials, Acta Mech. 64 (1986) 77–87.

[61] C. Thornton, D.J. Barnes, Computer simulated deformation of compact granular assemblies, Acta Mech. 64 (1986) 45–61.